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Notes on Noncommutative Instantons

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ABSTRACT: We study in detail the ADHM construction of $U(N)$ instantons on noncommutative Euclidean space-time $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ and noncommutative space $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}$. We point out that the completeness condition in the ADHM construction could be invalidated in certain circumstances. When this happens, regular instanton configuration may not exist even if the ADHM constraints are satisfied. Some of the existing solutions in the literature indeed violate the completeness condition and hence are not correct. We present alternative solutions for these cases in noncommutative space-time. However, it turns out that in the case of $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}$ there is a conceptual obstacle in finding non-singular instanton configurations. We also give a simple general argument based on the Corrigan's identity that the topological charge of noncommutative regular instantons on $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ is always an integer. Our results are different from previous ones which reported non-integral topological charge.

KEYWORDS: Noncommutative Geometry, Solitons Monopoles and Instantons.

1. Introduction

There has been a lot of interest in gauge theories on noncommutative spaces. One of the reasons for this interest is the natural appearance of noncommutativity $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ in the framework of string theory and D-branes [1–4]. Noncommutative gauge theories are also fascinating on their own right [5], mostly due to a mixing between the infrared (IR) and the ultraviolet (UV) degrees of freedom discovered in [6]. While UV/IR mixing arises at the perturbative level, it has been suggested that a resolution of it may lie at the nonperturbative level by resumming nonplanar diagrams at all loops [6, 8, 7, 9]. In gauge theories, one would also need to take into account the effects of noncommutative instantons on the IR physics [10–12].

Noncommutative instantons play an essential role in the understanding of the nonperturbative physics of noncommutative gauge theories. Employing the formalism Atiyah, Drinfeld, Hitchin and Manin (ADHM) [13], Nekrasov and Schwarz constructed [14] the first instanton on noncommutative $\mathbf{R}^4$. They showed that the ADHM constraints are modified by the noncommutativity. Solving the ADHM constraints, they obtained an anti-selfdual configuration whose singularity at short distances is regulated by the noncommutative scale. Following this pioneering work, the role of the projectors in the noncommutative ADHM construction was then clarified in a series of papers by Furuuchi [15–17]. Other related works appeared in [18–22,24,23]

As will be discussed in §3 two important elements in the ADHM construction are the *factorization* condition (3.15) and the *completeness* relation (3.16). It is well known that the factorization condition amounts to famous ADHM constraints. As it was shown in the original work of Nekrasov and Schwarz [14], the same is true in the noncommutative case and the ADHM constraints are modified in a relatively simple manner when noncommutativity is turned on. Solving the noncommutative ADHM constraints has been the focus in the literature since the work of [14]. However the completeness condition has not received the deserved attention so far. The completeness relation is the statement that the sum of the two projectors that one can construct out of the ADHM data spans the whole Hilbert space. While it is always guaranteed in the commutative case, it is no longer so when one turns on noncommutativity. The breakdown of this completeness relation is closely related to the fact that some states are projected out in the construction of the non-singular noncommutative instantons. Apparently this subtlety seems to have been overlooked. In §3.2 We analyse the completeness relation in detail to remedy this situation. In particular, it turns out that in the case of noncommutative space with commutative time there is a conceptual obstacle in finding non-singular instanton configurations. Hence, we have a strong conjecture that there are no instanton effects in noncommutative 4D gauge theories with commutative time.

It is somewhat reassuring to know that one can always find non-singular instanton solutions in noncommutative space-time. And it is also known that these instantons give non-trivial contributions to the path integral, [25, 11]. However, the meaning of quantum field theories with noncommutative time is not clear [26, 27]. The fact that we are not able to find regular noncommutative instantons in gauge theories with commutative time is very puzzling. This problem deserves further study from gauge and string theory perspectives.

The paper is organized as follows. In §2 we give a basic review of the properties of noncommutative space-time. In particular we consider two cases: (1) the noncommutative Euclidean space-time $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ with selfdual θ (SD-θ), and (2) the noncommutative space $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}$.

In §3 we review and discuss the ADHM construction of noncommutative instantons. In §3.2 we focus on discussing the factorization condition and the completeness relation. We show that the factorization condition amounts to the modified ADHM constraints and we find that solutions of the ADHM constraints do not necessarily guarantee the completeness relation. It is an independent condition and we give a necessary and sufficient condition (3.36) for the completeness relation to be valid. In §3.3, we use the Corrigan's identity to give a general argument that the topological charge of a noncommutative instanton is always an integer, $|Q| = k$.

We then turn our attention to explicit examples to illustrate these points. In §4 the construction of single $U(1)$ instanton configurations on $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ and $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}$ is studied. We show that anti-selfdual (ASD) instantons exist on $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ with SD-θ, but not the SD-instantons. For the case of $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}$, we find that the completeness relation is violated and no regular instanton can be constructed. The same is true for higher $U(N)$. In §5 we study the ASD $U(1)$ two-instanton solution. The topological charge is computed directly and is found to be an integer, $Q = -2$. Our result is different from a previous analysis [20] which reported a non-integral topological charge.

In §6 we construct the nonabelian ASD and SD instanton on SD-$\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$. For the case of ASD instanton, we find that the existing ansatz in the literature does not satisfy the completeness relation. Quite amazingly, an alternative ansatz (6.18) can be found, which does not contain an overall factor of the shift operator $u^\dagger$ as in the original ansatz (6.15), but satisfies the completeness relation and leads to a regular instanton solution. The topological charge of the ASD 1-instanton solution is explicitly computed and found to be equal to minus one. For the case of SD-instanton, the solution is regular and there is no need to introduce any projector. We also note that at large distances (or in the small noncommutativity limit) the SD/SD instanton approaches the regular-gauge BPST instanton, while the ASD/SD instanton tends to the singular-gauge BPST anti-instanton.

2. Noncommutative $\mathbf{R^4}$

We will work in flat Euclidean space-time $\mathbf{R^4}$ with noncommutative coordinates x^m which satisfy the commutation relations

$$[x^m, x^n] = i\theta^{mn} \ , \tag{2.1}$$

where $m, n = 1, 2, 3, 4$ are the Euclidean Lorentz indices and θ^{mn} is an antisymmetric real constant matrix. Using Euclidean space-time rotations, θ^{mn} can be always brought to the form

$$\theta^{mn} = \begin{pmatrix} 0 & \theta^{12} & 0 & 0 \\ -\theta^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^{34} \\ 0 & 0 & -\theta^{34} & 0 \end{pmatrix} . \tag{2.2}$$

In terms of complex coordinates

$$\begin{aligned} z_1 &= x_2 + ix_1 \ , \quad \bar{z}_1 = x_2 - ix_1 \ , \\ z_2 &= x_4 + ix_3 \ , \quad \bar{z}_2 = x_4 - ix_3 \ , \end{aligned} \tag{2.3}$$

the commutation relations (2.1) take the form

$$\begin{aligned} [z_1, \bar{z}_1] &= -2\theta^{12} \ , \quad [z_i, z_j] = 0 \ , \\ [z_2, \bar{z}_2] &= -2\theta^{34} \ , \quad [z_i, \bar{z}_{j\neq i}] = 0 \ , \end{aligned} \tag{2.4}$$

where $i, j = 1, 2$ denote the indices for the complex coordinates.

There are three important cases to consider:

- When $\theta^{12} = 0 = \theta^{34}$ all the commutators vanish giving the ordinary commutative $\mathbf{R^4}$. The corresponding world-volume gauge theory is the commutative gauge theory, and instanton solutions are given by the standard ADHM construction [13, 28–31]. Recent reviews and applications of the ADHM calculus can be found in [32, 33].

- When either θ^{12} or θ^{34} vanishes, the matrix θ^{mn} is of rank-two. This case corresponds to the direct product of the ordinary commutative 2-dimensional space with the noncommutative 2-dimensional space, $\mathbf{R^2_{\rm NC}} \times \mathbf{R^2}$. For definiteness we set here $\theta^{34} = 0$ and introduce the notation $\theta^{12} \equiv -\zeta/2$ in such a way that

$$[z_1, \bar{z}_1] = -\zeta \ , \quad [z_2, \bar{z}_2] = 0 \ , \quad [z_i, z_j] = 0 \ . \tag{2.5}$$

  Physical applications of this situation involve world-volume gauge theories with noncommutative space and commutative time.[1]

[1] It easily follows from (2.2) that the *general* description of noncommutative 3-dimensional space and commutative time is given by $\mathbf{R^2_{\rm NC}} \times \mathbf{R^2}$.

- A rank-four matrix θ^{mn} (with $\theta^{12} \neq 0$ and $\theta^{34} \neq 0$) generates the noncommutative Euclidean space-time $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}} = \mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}_{\mathrm{NC}}$. The corresponding world-volume gauge theory has noncommutative (Euclidean) time. Since both components of θ are non-vanishing, they can be made equal, $\theta^{12} = \theta^{34} \equiv -\zeta/4$, with appropriate rescalings of the four coordinates x^m and, if necessary, a parity transformation.[2] Equations (2.4) become

$$[z_i, \bar{z}_j] = -\frac{\zeta}{2}\delta_{ij} \ , \quad [z_i, z_j] = 0 \ . \tag{2.6}$$

In fact, the condition $\theta^{12} = \theta^{34}$ gives a selfdual (SD) theta, $\frac{1}{2}\epsilon^{mnkl}\theta_{kl} = \theta^{mn}$, while the anti-selfdual (ASD) theta, $\frac{1}{2}\epsilon^{mnkl}\theta_{kl} = -\theta^{mn}$, corresponds to $\theta^{12} = -\theta^{34}$. From now on when discussing the $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ case we will assume the space-time rescaling leading to the SD-theta, (2.6).[3]

The commutation relations (2.5) and (2.6) imply that the space-time coordinates x^m should be thought of as operators acting on a Hilbert space. The operator language and its Hilbert space representation will be discussed below in §2.1. In the following sections we will be concerned with constructing instanton configurations as the solutions to the operator-valued (anti)-selfduality equations (3.3). Following Nekrasov, Schwarz and Furuuchi, [14–17] we will use the operator-valued ADHM construction to determine the solutions of (3.3). In items 2 and 3 above we chose our notation in such a way that $\theta^{12}+\theta^{34} = \frac{1}{2}\zeta$, which will allow us to treat both noncommutative cases simultaneously. In §3 and §4 we will review the ADHM construction of instantons and point out important subtleties specific to noncommutative cases. For the case of noncommutative space-time the regular instanton solutions with integer topological charges can always be found, as will be described below. However, in the case of noncommutative space there is a conceptual obstacle in finding non-singular instanton configurations.

In order to discuss instanton effects in gauge theories on $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ we require a dictionary between the operator-valued ADHM instanton expressions and the ordinary c-number functions which are used for the functional integral representation of noncommutative gauge theories. This dictionary is provided by the map between the operators and the operator symbols outlined in §2.2.

In the semiclassical approximation to noncommutative gauge theories one expands the action around the minima of the Euclidean action – the noncommutative c-number-instantons – and integrates over the c-number-fluctuations around the instantons. The single-instanton

[2] Physics of noncommutative space-time is determined by θ^{mn} and certainly is not invariant under dilatations and parity transformations. However the general case can be always recovered from the simple case $\theta^{12} = \theta^{34}$ via opposite rescalings.

[3] The ASD-theta can be obtained from this by a parity transformation of two coordinates. This would also change the sign of the selfduality of the instanton configurations in the world-volume gauge theory.

integration measure can be in principle determined following the standard commutative instanton analysis of [38,39] by calculating Jacobians for the change of the integration variables from the fields to the instanton collective coordinates. The ADHM measure for 2 noncommutative $U(1)$ instantons was derived in [19]. The general all-orders in the instanton number ADHM supersymmetric measure was written down in Refs. [25,11].

2.1. Operator language

A Hilbert space representation for the noncommutative geometry (2.5) or (2.6) can be easily constructed by using complex variables (2.3) and realizing z and $\bar{z}$ as creation and annihilation operators in the Fock space for simple harmonic oscillators (SHO). The fields in a noncommutative gauge theory are described by functions of $z_1, \bar{z}_1, z_2, \bar{z}_2$. In the case of $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}$, the arguments z_2 and $\bar{z}_2$ are ordinary c-number coordinates, while z_1 and $\bar{z}_1$ are the creation and annihilation operators of a single SHO, $a^\dagger = z_1/\sqrt{\zeta}$ and $a = \bar{z}_1/\sqrt{\zeta}$.

The noncommutative space-time $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}} = \mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}_{\mathrm{NC}}$ requires two oscillators. The SHO Fock space $\mathcal{H}$ is spanned by the basis $|n_1, n_2\rangle$ with $n_1, n_2 \geq 0$:

$$\begin{aligned} z_1|n_1,n_2\rangle &= \sqrt{\tfrac{\zeta}{2}}\,\sqrt{n_1+1}|n_1+1,n_2\rangle, \quad z_2|n_1,n_2\rangle = \sqrt{\tfrac{\zeta}{2}}\,\sqrt{n_2+1}|n_1,n_2+1\rangle, \\ \bar{z}_1|n_1,n_2\rangle &= \sqrt{\tfrac{\zeta}{2}}\,\sqrt{n_1}|n_1-1,n_2\rangle, \quad \bar{z}_2|n_1,n_2\rangle = \sqrt{\tfrac{\zeta}{2}}\,\sqrt{n_2}|n_1,n_2-1\rangle. \end{aligned} \tag{2.7}$$

Derivatives of a function $f(z_1, \bar{z}_1, z_2, \bar{z}_2)$ are defined by

$$\partial_i f = \frac{2}{\zeta}[\bar{z}_i, f] \ , \quad \bar{\partial}_i f = -\frac{2}{\zeta}[z_i, f] \ , \tag{2.8}$$

and satisfy the standard requirements for $f = z_j$ or $f = \bar{z}_j$, as well as the chain rule and the useful identity for the derivative of the inverse function:

$$\partial_i f^{-1} = -f^{-1}(\partial_i f)f^{-1} \ , \quad \bar{\partial}_i f^{-1} = -f^{-1}(\bar{\partial}_i f)f^{-1} \ . \tag{2.9}$$

It is convenient to introduce differentials dz_i and $d\bar{z}_i$, which commute with $z, \bar{z}$'s and anticommute with each other, $dz_i d\bar{z}_j = -d\bar{z}_j dz_i$. We also introduce the (anti-)holomorphic exterior derivatives $\partial, \bar{\partial}$ and the total exterior derivative d,

$$d = \partial + \bar{\partial} \ , \quad \partial = dz_i \partial_i \ , \quad \bar{\partial} = d\bar{z}_i \bar{\partial}_i \ , \tag{2.10}$$

which satisfy

$$\partial^2 = \bar{\partial}^2 = 0 \ , \quad \partial\bar{\partial} = -\bar{\partial}\partial \ , \quad d^2 = 0 \ . \tag{2.11}$$

Nonholomorphic derivatives with respect to x^m are defined via

$$dx^m \frac{\partial}{\partial x^m} = d = \partial + \bar{\partial} = dz_i \partial_i + d\bar{z}_i \bar{\partial}_i \ , \tag{2.12}$$

and are explicitly given by

$$\begin{aligned} \partial \partial x^1 &= i(\partial_1 - \bar{\partial}_1) \ , \quad \partial \partial x^3 = i(\partial_2 - \bar{\partial}_2) \ , \\ \partial \partial x^2 &= \partial_1 + \bar{\partial}_1 \ , \quad \partial \partial x^4 = \partial_2 + \bar{\partial}_2 \ . \end{aligned} \tag{2.13}$$

We also note that $dz_1 d\bar{z}_1 dz_2 d\bar{z}_2 = -4 \ d^4x$.

In the complex basis the condition of selfduality can be easily formulated. A real 2-form F

$$F = (a_1 \ dz_1 d\bar{z}_1 - a_2 \ dz_2 d\bar{z}_2) + b \ dz_1 d\bar{z}_2 + \bar{b} \ dz_2 d\bar{z}_1 + c \ dz_1 dz_2 + \bar{c} \ d\bar{z}_2 d\bar{z}_1 \tag{2.14}$$

is anti-selfdual iff $a_1 = a_2$ and $c = \bar{c} = 0$. (F is selfdual iff $a_1 = -a_2$ and $b = \bar{b} = 0$.)

The integral on $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ is defined by the operator trace,

$$\int d^4x = (2\pi)^2 \sqrt{\det\theta} \ \mathrm{Tr} = (\frac{\zeta\pi}{2})^2 \ \mathrm{Tr} \ . \tag{2.15}$$

In the ordinary commutative case the integral of the total derivative of a regular function vanishes when the function falls off fast enough at infinity. In the noncommutative case the integral of a total derivative is the trace of a commutator. It vanishes only when the trace of each term in the commutator is individually well-defined (finite). Hence, similarly to the commutative case, we see that $\int d^4x \ \partial_m K^m$ can receive non-vanishing contributions from the 'boundary of integration'.

2.2. Operator symbols

Operator symbols are ordinary c-number functions which provide an alternative to the operator language. Using operator symbols, the fields of a noncommutative gauge theory are viewed as ordinary c-number functions which are multiplied using the star-product:

$$(\Phi * \Psi)(x) \equiv \Phi(x) e^{i2\theta^{mn} \overleftarrow{\partial}_m \overrightarrow{\partial}_n} \Psi(x) \ , \tag{2.16}$$

where here ∂_m denotes $\partial/\partial x^m$. This is achieved for each of the $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}}$ factors in $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}} \times \mathbf{R}^{\mathbf{2}}$ or $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}} = \mathrm{NC}\mathbf{R}^{\mathbf{2}} \times \mathbf{R}^{\mathbf{2}}_{\mathrm{NC}}$ via a one-to-one map Ω from the operators on $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}}$ to the c-number operator symbols on $\mathbf{R}^{\mathbf{2}}$. This map can be defined [16] via an inverse normal ordered Fourier transform,

$$\Omega \ : \ \hat{\phi}(\hat{x}) \ \rightarrow \ \Phi(x) \equiv \int d^2k (2\pi)^2 e^{ikx} \int d^2\hat{x} \ : e^{-ik\hat{x}} : \ \hat{\phi}(\hat{x}) \ , \tag{2.17}$$

where hats denote operator quantities, $\int d^2x$ is the normalized operator trace $2\pi|\theta|$ Tr, and $: e^{ik\hat{x}} :$ stands for the normal ordered exponent. This expression for the operator symbol is particularly transparent in the coherent state basis:

$$\Omega \ : \ \hat{\phi}(\hat{z}, \hat{\bar{z}}) \ \rightarrow \ \Phi(z, \bar{z}) = \langle z|\hat{\phi}(\hat{z}, \hat{\bar{z}})|\bar{z}\rangle \ , \tag{2.18}$$

where $|\bar{z}\rangle$ and $\langle z|$ denote normalized coherent states,

$$\hat{\bar{z}}|\bar{z}\rangle = \bar{z}|\bar{z}\rangle \ , \quad \langle z|\hat{z} = \langle z|z \ , \quad \langle z|\bar{z}\rangle = 1 \ , \tag{2.19}$$

and

$$\langle z| : e^{ik\hat{x}} : |\bar{z}\rangle = e^{ikx} \ . \tag{2.20}$$

Useful examples of this correspondence include the expressions for the operator symbols of the Fock states projectors,

$$\Omega \ : \ |m\rangle\langle n| \ \rightarrow \ 1\sqrt{m!}\left(z\sqrt{2|\theta|}\right)^m 1\sqrt{n!}\left(\bar{z}\sqrt{2|\theta|}\right)^n \ e^{-2z\bar{z}/|\theta|} \ , \tag{2.21}$$

and the coherent states projectors,

$$\Omega \ : \ |\bar{w}\rangle\langle w| \ \rightarrow \ e^{-2(z-w)(\bar{z}-\bar{w})/|\theta|} \ . \tag{2.22}$$

Using the dictionary above outlined, we can easily turn operator-valued expressions into ordinary functions. As already mentioned, this is an important requirement for the functional integral representation of noncommutative gauge theories, which uses ordinary functions (albeit multiplied using the star-product). The map between the operators and the operator symbols provides the link between the ADHM instantons and their semiclassical contributions.

3. ADHM construction of instantons

In this section we describe the construction of instantons due to Atiyah, Drinfeld, Hitchin and Manin (ADHM) [13], which was first applied to noncommutative gauge theories by Nekrasov and Schwarz [14]. The commutative ADHM construction was also discussed in Refs. [28–31, 36, 34, 35, 32, 33]. Here we follow the $U(N)$ formalism of Refs. [32, 33].

Consider a pure $U(N)$ gauge theory formulated on a generic noncommutative Euclidean space (it can be $\mathbf{R}^{\mathbf{4}}_{\rm NC}$ or $\mathbf{R}^{\mathbf{2}}_{\rm NC} \times \mathbf{R}^{\mathbf{2}}$). The action in the operator language is given by

$$S[A] = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}_N \, F_{mn}F_{mn} \ , \tag{3.1}$$

where F_{mn} is the field strength $F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n]$. The topological charge Q is defined by

$$Q = -\frac{1}{16\pi^2} \int d^4x \, \mathrm{Tr}_N F_{mn} \tilde{F}_{mn} \equiv -\frac{1}{16\pi^2} \int d^4x \, \partial_m K^m \ , \tag{3.2}$$

where $\tilde{F}_{mn} = \frac{1}{2}\epsilon_{mnkl} F_{kl}$ and we used the well-known fact that $\mathrm{Tr}_N F_{mn} \tilde{F}_{mn}$ can be written as a total derivative. In the ordinary commutative case the topological charge is an integer equal to the winding number of the map $S^3 \to S^3$. Here the first S^3 is the boundary at infinity of the space-time $\mathbf{R^4}$, and the second S^3 is an $SU(2)$ subgroup of the gauge group $U(N)$. The k-instanton configuration is then defined as the general solution of the (anti)-selfduality equation

$$F^{mn} = \pm\tfrac{1}{2}\epsilon^{mnkl} F_{kl} \tag{3.3}$$

in the topological sector $Q = k$. Instantons automatically solve the nonabelian Maxwell equations thanks to the Jacobi identity and, hence, give local minima of the Euclidean action (3.1):

$$S_{\mathrm{inst}} = \frac{8\pi^2 |k|}{g^2} \ . \tag{3.4}$$

We will see that the topological charge calculated on instanton configurations in noncommutative $U(N)$ gauge theories is still an integer[4] for all values of $N \geq 1$. For $N \geq 2$ one might conjecture that this result still has topological origins: first, express Q as an integral of the total derivative $\partial_m K^m$, and, second, use the c-number operator symbols to evaluate K^m. With an additional assumption that there are no singularities in K^m at finite values of x, one would conclude that Q receives contributions only from the sphere S^3 at spatial infinity, where noncommutativity is irrelevant and Q is an integer. This argument, however, does not explain why Q is an integer for $U(1)$.

When discussing instantons it is very convenient to introduce a *quaternionic* notion for the four-dimensional Euclidean space-time indices

$$x_{[2]\times[2]} \equiv x_{\alpha\dot{\alpha}} = x_n \sigma_{n\alpha\dot{\alpha}} \ , \qquad \bar{x}_{[2]\times[2]} \equiv \bar{x}^{\dot{\alpha}\alpha} = x_n \bar{\sigma}_n^{\dot{\alpha}\alpha} \ , \tag{3.5}$$

[4] There is some confusion in the literature concerning this issue. In [16] Furuuchi argued that the topological charge of noncommutative $U(1)$ instantons must be integer, however the authors of [20] have calculated Q explicitly for the 2-instanton in $U(1)$ and got a non-integer result. They also obtained non-integer expressions for Q using a $U(2)$ instanton. In addition, the authors of [23] showed that the noncommutative version of 't Hooft ansatz, previously introduced in [14], does not give a selfdual configuration, and, hence, its topological charge is not an integer. An alternative solution presented in [23] does have an integer Q, but at a cost of introducing a non-Hermitean field-strength. Below we will give a general argument that for all values of N the topological charge of the noncommutative $U(N)$ instanton is integer and, moreover, equal to the instanton number k. Then in the next section we also provide explicit calculations of Q at the 1- and 2-instanton level, finding integer results.

where $\sigma_{n\alpha\dot\alpha}$ are the components of four 2×2 matrices $\sigma_n = (i\vec{\tau}, 1_{[2]\times[2]})$, and τ^c, $c = 1,2,3$ are the three Pauli matrices. In addition we define the Hermitian conjugate matrices $\bar\sigma_n = \sigma_n^\dagger = (-i\vec\tau, 1_{[2]\times[2]})$ with components $\bar\sigma_n^{\dot\alpha\alpha}$.[5] In terms of the complex coordinates (2.3) we have

$$x_{\alpha\dot\alpha} = \begin{pmatrix} z_2 & z_1 \\ -\bar z_1 & \bar z_2 \end{pmatrix} \ , \qquad \bar x^{\dot\alpha\alpha} = \begin{pmatrix} \bar z_2 & -z_1 \\ \bar z_1 & z_2 \end{pmatrix} \ . \tag{3.6}$$

The tensor

$$\sigma_{mn\,\alpha}{}^{\beta} \equiv \tfrac14(\sigma_{m\alpha\dot\alpha}\bar\sigma_n^{\dot\alpha\beta} - \sigma_{n\alpha\dot\alpha}\bar\sigma_m^{\dot\alpha\beta}) = \tfrac12 i\eta^a_{mn}\tau^a \ , \tag{3.7}$$

is selfdual, $\frac12\epsilon_{mnkl}\sigma_{kl} = \sigma_{mn}$, while

$$\bar\sigma_{mn}{}^{\dot\alpha}{}_{\dot\beta} \equiv \tfrac14(\bar\sigma_m^{\dot\alpha\alpha}\sigma_{n\alpha\dot\beta} - \bar\sigma_n^{\dot\alpha\alpha}\sigma_{m\alpha\dot\beta}) = \tfrac12 i\bar\eta^a_{mn}\tau^a \ , \tag{3.8}$$

is anti-selfdual, $\frac12\epsilon_{mnkl}\bar\sigma_{kl} = -\bar\sigma_{mn}$. Here η^a_{mn} and $\bar\eta^a_{mn}$ are the standard 't Hooft η-symbols.

3.1. The gauge field and the field strength

The basic object in the ADHM construction of selfdual k-instantons (SD-instantons) is the $(N+2k)\times 2k$ complex-valued matrix $\Delta_{[N+2k]\times[2k]}$ which is taken to be linear in the space-time variable x_n:[6]

$$\text{SD instanton}: \quad \Delta_{[N+2k]\times[2k]}(x) \equiv \Delta_{[N+2k]\times[k]\times[2]}(x) = a_{[N+2k]\times[k]\times[2]} + b_{[N+2k]\times[k]\times\underline{[2]}}\ x_{\underline{[2]}\times[2]} \ . \tag{3.9}$$

Here we have represented the $[2k]$ index set as a product of two index sets $[k]\times[2]$ and used a quaternionic representation of x_n as in (3.5). By counting the number of bosonic zero modes, we will soon verify that k in Eq. (3.9) is indeed the instanton number, while N is the number of colours of the gauge group $U(N)$. We further choose a representation in which b assumes a simple canonical form [28]:

$$b_{[N+2k]\times[2k]} = \begin{pmatrix} 0_{[N]\times[2k]} \\ 1_{[2k]\times[2k]} \end{pmatrix} \ , \qquad a_{[N+2k]\times[2k]} = \begin{pmatrix} w_{[N]\times[2k]} \\ a'_{[2k]\times[2k]} \end{pmatrix} \ . \tag{3.10}$$

As discussed below, the complex-valued constant matrix a in Eq. (3.9) constitutes a (highly overcomplete) set of collective coordinates on the instanton moduli space $\mathfrak{M}_k$.

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[5] Notice that the spinor indices $\alpha, \dot\alpha = 1,2$ are raised and lowered with the ϵ-tensor: $\bar x^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}x_{\beta\dot\beta}$.

[6] For clarity, in this section, we will occasionally show matrix sizes explicitly, *e.g.* the $U(N)$ gauge field will be denoted $A_{n[N]\times[N]}$. To represent matrix multiplication in this notation we will underline contracted indices: $(AB)_{[a]\times[c]} = \ A_{[a]\times\underline{[b]}}\ B_{\underline{[b]}\times[c]}$. Also we adopt the shorthand $X_{[m}Y_{n]} = X_mY_n - X_nY_m$.

The matrix Δ used for the construction of the anti-selfdual instanton (ASD instanton) is given by

$$\text{ASD instanton}: \quad \Delta_{[N+2k]\times[2k]}(x) \equiv \Delta_{[N+2k]\times[k]\times[2]}(x) = a_{[N+2k]\times[k]\times[2]} + b_{[N+2k]\times[k]\times\underline{[2]}}\ \bar{x}_{\underline{[2]}\times[2]}\ . \quad (3.11)$$

It follows from the definitions that for the SD instanton $\partial_n\Delta = b\sigma_n$, whereas for the ASD instanton $\partial_n\Delta = b\bar{\sigma}_n$.

For generic x, the null-space of the Hermitian conjugate matrix $\bar{\Delta}(x)$ is N-dimensional, as it has N fewer rows than columns. The basis vectors for this null-space can be assembled into an $(N+2k)\times N$ dimensional complex-valued matrix $U(x)$,[7]

$$\bar{\Delta}_{[2k]\times\underline{[N+2k]}} U_{\underline{[N+2k]}\times[N]} = 0 = \bar{U}_{[N]\times\underline{[N+2k]}} \Delta_{\underline{[N+2k]}\times[2k]}\ , \tag{3.12}$$

where U is orthonormalized according to

$$\bar{U}_{[N]\times\underline{[N+2k]}} U_{\underline{[N+2k]}\times[N]} = 1_{[N]\times[N]}\ . \tag{3.13}$$

In turn, the classical ADHM gauge field A_n is constructed from U as follows:

$$A_{n[N]\times[N]} = \bar{U}_{[N]\times\underline{[N+2k]}} \partial_n U_{\underline{[N+2k]}\times[N]}\ . \tag{3.14}$$

Note first that for the special case $k=0$, the gauge configuration A_n defined by (3.14) is a "pure gauge" so that it automatically solves the selfduality equation (3.3) in the vacuum sector. The ADHM ansatz is that Eq. (3.14) continues to give a solution to Eq. (3.3), even for nonzero k. As we shall see, this requires two additional conditions. The first one is the so-called *factorization* condition:

$$\bar{\Delta}_{[2]\times[k]\times\underline{[N+2k]}} \Delta_{\underline{[N+2k]}\times[k]\times[2]} = 1_{[2]\times[2]} f^{-1}_{[k]\times[k]}\ , \tag{3.15}$$

where f is an arbitrary x-dependent $k\times k$ dimensional Hermitian matrix. The second condition is the so-called *completeness* relation:

$$1_{[N+2k]\times[N+2k]} = \Delta_{[N+2k]\times\underline{[k]}\times\underline{[2]}} f_{\underline{[k]}\times\underline{[k]}} \bar{\Delta}_{\underline{[2]}\times\underline{[k]}\times[N+2k]} + U_{[N+2k]\times\underline{[N]}} \bar{U}_{\underline{[N]}\times[N+2k]}\ . \tag{3.16}$$

Note that both terms on the right hand side of (3.16) are Hermitian projection operators

$$P_{[N+2k]\times[N+2k]} \equiv \Delta_{[N+2k]\times\underline{[k]}\times\underline{[2]}} f_{\underline{[k]}\times\underline{[k]}} \bar{\Delta}_{\underline{[2]}\times\underline{[k]}\times[N+2k]}\ , \quad \mathcal{P}_{[N+2k]\times[N+2k]} \equiv U_{[N+2k]\times\underline{[N]}} \bar{U}_{\underline{[N]}\times[N+2k]}\ . \tag{3.17}$$

Both conditions, (3.15) and (3.16), will be investigated below in §3.2. With the above relations together with integrations by parts, the expression for the field strength F_{mn} for the SD instanton may be elaborated as follows:

$$\begin{aligned} F_{mn} &\equiv \partial_{[m}A_{n]} + A_{[m}A_{n]} = \partial_{[m}(\bar{U}\partial_{n]}U) + (\bar{U}\partial_{[m}U)(\bar{U}\partial_{n]}U) = \partial_{[m}\bar{U}(1-U\bar{U})\partial_{n]}U \\ &= \partial_{[m}\bar{U}\Delta f\bar{\Delta}\partial_{n]}U = \bar{U}\partial_{[m}\Delta f\partial_{n]}\bar{\Delta}U = \bar{U}b\sigma_{[m}\bar{\sigma}_{n]}f\bar{b}U = 4\bar{U}b\sigma_{mn}f\bar{b}U\ . \end{aligned} \tag{3.18}$$

[7]Throughout this, and other sections, an overbar means hermitian conjugation: $\bar{\Delta} \equiv \Delta^\dagger$.

Selfduality of the field strength then follows automatically from (3.7).

For the ASD instanton field strength we get:

$$F_{mn} = \bar{U}\partial_{[m}\Delta f \partial_{n]}\bar{\Delta}U = \bar{U}b\bar{\sigma}_{[m}\sigma_{n]}f\bar{b}U = 4\bar{U}b\bar{\sigma}_{mn}f\bar{b}U \ , \tag{3.19}$$

which is anti-selfdual due to (3.8).

3.2. Constraints and projectors

We have seen that the ADHM construction for $U(N)$ makes essential use of matrices of various sizes: $(N+2k)\times N$ matrices U, $(N+2k)\times 2k$ matrices Δ, a and b, $k\times k$ matrices f, and 2×2 matrices $\sigma_{n\alpha\dot{\alpha}}$, $\bar{\sigma}_n^{\dot{\alpha}\alpha}$, $x_{\alpha\dot{\alpha}}$, *etc.* Accordingly, we introduce a variety of index assignments:

$$\begin{aligned}
\text{Instanton number indices } [k] :&\quad 1\le i,j,l,\ldots\le k\\
\text{Color indices } [N] :&\quad 1\le u,v,\ldots\le N\\
\text{ADHM indices } [N+2k] :&\quad 1\le \lambda,\mu,\ldots\le N+2k\\
\text{Quaternionic (Weyl) indices } [2] :&\quad \alpha,\beta,\dot{\alpha},\dot{\beta},\ldots = 1,2\\
\text{Lorentz indices } [4] :&\quad m,n,\ldots = 0,1,2,3
\end{aligned}$$

No extra notation is required for the $2k$ dimensional column index attached to Δ, a and b, since it can be factored as $[2k]=[k]\times[2]=j\dot{\beta}$, *etc.*, as in Eq. (3.9). With these index conventions, Eq. (3.9) for the SD instanton reads

$$\text{SD instanton}:\quad \Delta_{\lambda i\dot{\alpha}}(x) = a_{\lambda i\dot{\alpha}} + b^{\alpha}_{\lambda i}x_{\alpha\dot{\alpha}} \ ,\qquad \bar{\Delta}_i^{\dot{\alpha}\lambda}(x) = \bar{a}_i^{\dot{\alpha}\lambda} + \bar{x}^{\dot{\alpha}\alpha}\,\bar{b}^{\lambda}_{\alpha i} \ , \tag{3.20}$$

while the factorization condition (3.15) becomes

$$\bar{\Delta}_i^{\dot{\beta}\lambda}\Delta_{\lambda j\dot{\alpha}} = \delta^{\dot{\beta}}{}_{\dot{\alpha}}(f^{-1})_{ij} \ . \tag{3.21}$$

Notice that by definition $\bar{\Delta}_i^{\dot{\alpha}\lambda}\equiv(\Delta_{\lambda i\dot{\alpha}})^*$.

Factorization condition and ADHM constraints

We can make the canonical form (3.10) a little more explicit with a convenient representation of the index set $[N+2k]$. We decompose each ADHM index $\lambda\in[N+2k]$ into[8]

$$\lambda = u + i\alpha \ ,\quad 1\le u\le N \ ,\quad 1\le i\le k \ ,\quad \alpha=1,2 \ . \tag{3.22}$$

[8]The Weyl index α in this decomposition is raised and lowered with the ϵ tensor as always, whereas for the $[N]$ and $[k]$ indices u and i there is no distinction between upper and lower indices.

In other words, the top $N\times 2k$ submatrices in Eq. (3.10) have rows indexed by $u\in[N]$, whereas the bottom $2k\times 2k$ submatrices have rows indexed by the pair $i\alpha\in[k]\times[2]$. Equation (3.10) then becomes

$$a_{\lambda j\dot\alpha}=a_{(u+i\alpha)j\dot\alpha}=\begin{pmatrix} w_{uj\dot\alpha}\\ (a'_{\alpha\dot\alpha})_{ij}\end{pmatrix}\quad , \tag{3.23a}$$

$$\bar a_j^{\dot\alpha\lambda}=\bar a_j^{\dot\alpha(u+i\alpha)}=\begin{pmatrix}\bar w_{ju}^{\dot\alpha} & (\bar a'^{\dot\alpha\alpha})_{ji}\end{pmatrix}\quad , \tag{3.23b}$$

$$b_{\lambda j}^{\beta}=b_{(u+i\alpha)j}^{\beta}=\begin{pmatrix}0\\ \delta_\alpha{}^\beta\delta_{ij}\end{pmatrix}\quad , \tag{3.23c}$$

$$\bar b_{\beta j}^{\lambda}=\bar b_{\beta j}^{u+i\alpha}=\begin{pmatrix}0 & \delta_\beta{}^\alpha\,\delta_{ji}\end{pmatrix}\quad . \tag{3.23d}$$

Combining Eqs. (3.20)-(3.23d), and noting that $f_{ij}(x)$ is arbitrary, one extracts the x-independent conditions on the matrix a:

$$\text{SD instanton}:\quad \tau^{c\,\dot\alpha}{}_{\dot\beta}(\bar a^{\dot\beta}a_{\dot\alpha})_{ij}-\delta_{ij}\bar\eta^c_{mn}\theta^{mn}=0 \tag{3.24a}$$

$$(a'_n)^\dagger=a'_n\ . \tag{3.24b}$$

In Eq. (3.24a) there are three separate equations since we have contracted $\bar a^{\dot\beta}a_{\dot\alpha}$ with any of the three Pauli matrices, while in Eq. (3.24b) we have decomposed $(a'_{\alpha\dot\alpha})_{ij}$ and $(\bar a'^{\dot\alpha\alpha})_{ij}$ in our usual quaternionic basis of spin matrices:

$$(a'_{\alpha\dot\alpha})_{ij}=(a'_n)_{ij}\sigma_{n\alpha\dot\alpha}\ ,\quad (\bar a'^{\dot\alpha\alpha})_{ij}=(a'_n)_{ij}\bar\sigma_n^{\dot\alpha\alpha}\ . \tag{3.25}$$

The three conditions (3.24a) are the modified ADHM constraints for the SD instanton. When $\theta^{mn}=0$ Eqs. (3.24a) give the standard commutative ADHM constraints [28, 29]. When noncommutativity is present, the SD instanton constraints are modified by the ASD component of θ. Thus, the ADHM constraints for the SD instanton in the SD-θ background on $\mathbf{R}^4_{\mathrm{NC}}$ are unmodified,

$$\tau^{c\,\dot\alpha}{}_{\dot\beta}(\bar a^{\dot\beta}a_{\dot\alpha})_{ij}=0\ . \tag{3.26}$$

At the same time, the constraints for the SD instanton in noncommutative space $\mathbf{R}^2_{\mathrm{NC}}\times\mathbf{R}^2$ are modified,

$$\tau^{c\,\dot\alpha}{}_{\dot\beta}(\bar a^{\dot\beta}a_{\dot\alpha})_{ij}-\delta_{ij}\delta^{c3}\zeta=0\ . \tag{3.27}$$

The ASD instanton constraints follow from solving the same factorization condition (3.15) with the matrix Δ given by (3.11). In this case the ADHM constraints are modified by the SD component of θ,

$$\text{ASD instanton}:\quad \tau^{c\,\dot\alpha}{}_{\dot\beta}(\bar a^{\dot\beta}a_{\dot\alpha})_{ij}-\delta_{ij}\eta^c_{mn}\theta^{mn}=0 \tag{3.28a}$$

$$(a'_n)^\dagger=a'_n\ . \tag{3.28b}$$

From (3.28a) it follows that the constraints for the ASD instanton in the SD-θ background on $\mathbf{R}^4_{\mathrm{NC}}$, and for the ASD instanton in noncommutative space $\mathbf{R}^2_{\mathrm{NC}} \times \mathbf{R}^2$, are modified in the same way:

$$\tau^{c\,\dot{\alpha}}{}_{\dot{\beta}}(\bar{a}^{\dot{\beta}} a_{\dot{\alpha}})_{ij} - \delta_{ij}\delta^{c3}\zeta = 0 \ . \tag{3.29}$$

The ADHM constraints define a set of coupled quadratic conditions on the matrix elements of a which have to be solved in order to determine each SD k-instanton solution explicitly. The elements of the matrix a comprise the collective coordinates for the k-instanton gauge configuration. Clearly the number of independent such elements grows as k^2, even after accounting for the constraints. In contrast, the number of physical collective coordinates should equal $4kN$ which scales linearly with k. It follows that a constitutes a highly redundant set of parameters. Much of this redundancy can be eliminated by noting that the ADHM construction with b in the canonical form (3.10) is unaffected by x-independent transformations of the form:

$$\Delta_{[N+2k]\times[2k]} \to \begin{pmatrix} 1_{[N]\times[N]} & 0_{[2k]\times[N]} \\ 0_{[N]\times[2k]} & \mathcal{U}^\dagger_{[2k]\times[2k]} \end{pmatrix} \Delta_{[N+2k]\times[2k]} \mathcal{U}_{[2k]\times[2k]} \tag{3.30}$$

where $\mathcal{U}_{[2k]\times[2k]} = \mathfrak{U}_{ij}\delta^{\dot{\beta}}{}_{\dot{\alpha}}$ and $\mathfrak{U}_{ij} \in \mathrm{U}(k)$. In terms of w and a', this $U(k)$ symmetry transformation acts as

$$w_{ui\dot{\alpha}} \to w_{uj\dot{\alpha}}\mathfrak{U}_{ji} \ , \qquad (a'_{\alpha\dot{\alpha}})_{ij} \to \mathfrak{U}^\dagger_{ik}(a'_{\alpha\dot{\alpha}})_{kl}\mathfrak{U}_{lj} \ . \tag{3.31}$$

From now on, we will take the basic ADHM variables to be the complex quantities $w_{ui\dot{\alpha}}$, where $\bar{w}^{\dot{\alpha}}_{iu} \equiv (w_{ui\dot{\alpha}})^*$, and the four $k \times k$ Hermitian matrices a'_n. Now we can count the independent collective coordinate degrees of freedom of the ADHM k-instanton. The matrix $w_{ui\dot{\alpha}}$ contributes $4kN$ real degrees of freedom and Hermitian matrices a'_n give $4k^2$. The ADHM conditions (3.29) impose $3k^2$ real constraints, and modding out by the $U(k)$ symmetry group removes further k^2 degrees of freedom. In total we have precisely $4Nk$ real degrees of freedom left, which is precisely the expected number of independent k-instanton collective coordinates. Of these it is easy to identify four coordinates which correspond to instanton translations X_n:

$$\begin{aligned} \text{SD instanton :} \qquad & a_{[N+2k]\times[k]\times[2]} = b_{[N+2k]\times[k]\times\underline{[2]}}\ X_{\underline{[2]}\times[2]} \ , \\ \text{ASD instanton :} \qquad & a_{[N+2k]\times[k]\times[2]} = b_{[N+2k]\times[k]\times\underline{[2]}}\ \bar{X}_{\underline{[2]}\times[2]} \ . \end{aligned} \tag{3.32}$$

as is obvious from (3.9) and (3.11).

Completeness relation

We can now study the completeness relation (3.16). This relation is automatic in the standard commutative case, but we will point out that there are subtleties in the noncommutative case, where x itself is an operator. In the noncommutative case the Hermitian projector

$P = \Delta f \bar{\Delta}$ defined in (3.17) is an $[N+2k] \times [N+2k]$ matrix of operators on a Fock space $\mathcal{H}$

$$P : \mathcal{H}^{N+2k} \to P\mathcal{H}^{N+2k} \subset \mathcal{H}^{N+2k} . \tag{3.33}$$

We start by considering the eigenvalue problem for P. Since P is a Hermitian projection operator, $P^\dagger = P$ and $PP = P$, all its eigenvalues are either equal to zero or equal to one. Let $|\Psi^p\rangle$ and $|\Phi^r\rangle$ denote the normalized zero-mode and non-zero-mode eigenstates of P:

$$P|\Psi^p\rangle = 0 , \quad |\Psi^p\rangle \in \mathcal{H}^{N+2k} , \quad \langle \Psi^p|\Psi^q\rangle = \delta^{pq} , \tag{3.34a}$$

$$P|\Phi^r\rangle = |\Phi^r\rangle , \quad |\Phi^r\rangle \in \mathcal{H}^{N+2k} , \quad \langle \Phi^r|\Phi^s\rangle = \delta^{rs} . \tag{3.34b}$$

Now, since the set of all eigenstates of a Hermitian operator is complete we can write

$$\begin{aligned} 1_{[N+2k]\times[N+2k]} &= \sum_p |\Psi^p\rangle\langle\Psi^p| + \sum_r |\Phi^r\rangle\langle\Phi^r| \\ &= \sum_p |\Psi^p\rangle\langle\Psi^p| + \Delta_{[N+2k]\times\underline{[k]}\times\underline{[2]}} f_{\underline{[k]}\times\underline{[k]}} \bar{\Delta}_{\underline{[2]}\times\underline{[k]}\times[N+2k]} . \end{aligned} \tag{3.35}$$

The second line in (3.35) follows from the fact that all non-zero eigenvalues of $P = \Delta f \bar{\Delta}$ are equal to one.

The ADHM relation (3.16) will follow from (3.35) if and only if all the zero-mode eigenstates of P can be written as:

$$\Big\{|\Psi^p\rangle\Big\} = \Big\{U_{[N+2k]\times\underline{[N]}}|s_{\underline{[N]}}\rangle\Big\} , \tag{3.36}$$

where $|s_u\rangle$ are arbitrary normalized states in $\mathcal{H}$. If the requirement (3.36) holds, then $\sum_p |\Psi^p\rangle\langle\Psi^p| = U\bar{U}$ and the completeness relation (3.16) follows. If (3.36) does not hold, we cannot use (3.16) and the ADHM construction of (A)SD field strengths necessarily breaks down.

Note that the condition (3.36) is not automatic. While it is always true that a state $U|s\rangle \neq 0$ is necessarily a normalized zero-mode eigenstate of $P = \Delta f \bar{\Delta}$ (this follows from (3.12)-(3.13)), it is not generally correct to assume that each zero mode can be represented in this way.

In the following sections we will analyse (3.36) and (3.16) for various explicit noncommutative instanton solutions. It will follow from our considerations that the condition (3.36) is always invalidated when one attempts to construct regular instantons in gauge theories on $\mathbf{R}^{\mathbf{2}}_{\rm NC} \times \mathbf{R}^{\mathbf{2}}$, i.e. noncommutative instantons with commutative time do not exist. At the same time we will find that there is no obstacle in satisfying (3.36) in the ADHM construction on $\mathbf{R}^{\mathbf{4}}_{\rm NC}$.

3.3. Topological charge and Corrigan's identity

The topological charge of the SD ADHM k-instanton is given by

$$Q = -\frac{1}{16\pi^2} \int d^4x \, \mathrm{Tr}_N F_{mn} F_{mn} \ , \tag{3.37}$$

where the SD field strength is given by

$$F_{mn} = \bar{U} b \sigma_{[m} \bar{\sigma}_{n]} f \bar{b} U \ . \tag{3.38}$$

The integral in (3.37) can be evaluated in general thanks to a remarkable Corrigan's identity:

$$\mathrm{Tr}_N F_{mn} F_{mn} = \tfrac{1}{2} \partial_n \partial^n \mathrm{Tr} b \sigma_m (\mathcal{P} + 1) \bar{\sigma}_m b f \ , \tag{3.39}$$

where Tr means a trace over both $U(N)$ and ADHM indices and $\mathcal{P} = 1 - \Delta f \bar{\Delta}$ was defined in (3.17). The relation (3.39) was first derived in the commutative case in [30]. A brute-force proof of (3.39) in the commutative $SU(2)$ case which appeared in Appendix C2 of [34] can be directly applied to the noncommutative $U(N)$ construction.

Thanks to (3.39) Q is the integral of a total derivative, hence the straightforward way to evaluate it is to map the operator-valued expressions to the operator symbols and saturate the integral on the boundary. As always instanton configurations which are relevant to semiclassical functional integral applications are either regular or are gauge equivalent to the regular configurations. Thus, since (3.39) is gauge invariant due to Tr, we can assume that the expression $\frac{1}{2} \partial^n \mathrm{Tr} b \sigma_m (\mathcal{P} + 1) \bar{\sigma}_m b f$ contains no singularities at finite values of x. Hence, Q receives contributions solely from the boundary S^3 at infinity,

$$Q = \frac{1}{16\pi^2} \frac{1}{2} 2\pi^2 \cdot 2 \ \mathrm{Tr} \ b \sigma_m (\mathcal{P}_\infty + 1) \bar{\sigma}_m b \ = k \ . \tag{3.40}$$

In deriving (3.40) we used the following asymptotics:

$$\Delta \to bx \ , \quad f_{ij} \to 1x^2 \delta_{ij} \ , \quad \mathcal{P} \to 1 - b\bar{b} \equiv \mathcal{P}_\infty \ , \qquad \text{as } |x| \to \infty. \tag{3.41}$$

Thus, we conclude that the topological charge of the noncommutative SD ADHM k-instanton is always equal to k. An almost identical calculation for the ASD k-instanton gives $Q = -k$. It is remarkable that the fact that the topological charge is an integer and is equal to $\pm k$ is basically an algebraic statement encoded in the structure of the ADHM matrices even in the noncommutative case. In the following sections we will evaluate topological charges of some simple instanton solutions without making use of this powerful argument.

4. U(1) single-instanton solution

In this section we analyse in detail an explicit construction of the ASD single instanton solution in the noncommutative $U(1)$ gauge theory. Singular $U(1)$ instantons on $\mathbf{R}^4_{\mathrm{NC}}$ were first discussed in [14], and the regular solutions were constructed in [15, 16]. Here we will treat the two noncommutative backgrounds: (1) SD-θ on $\mathbf{R}^4_{\mathrm{NC}}$, and (2) $\mathbf{R}^2_{\mathrm{NC}} \times \mathbf{R}^2$ in parallel, and show that there is an obstacle in constructing a well-defined instanton on (2) – a general result valid for all $U(N)$ groups.

The ADHM matrix Δ for the ASD instanton (3.11) which satisfies the modified ADHM constraints (3.28b), (3.29) is given by:

$$\Delta = \begin{pmatrix} \sqrt{\zeta} & 0 \\ \bar{z}_2 - \bar{Z}_2 & -(z_1 - Z_1) \\ \bar{z}_1 - \bar{Z}_1 & z_2 - Z_2 \end{pmatrix} , \qquad \bar{\Delta} = \begin{pmatrix} \sqrt{\zeta} & z_2 - Z_2 & z_1 - Z_1 \\ 0 & -(\bar{z}_1 - \bar{Z}_1) & (\bar{z}_2 - \bar{Z}_2) \end{pmatrix} . \tag{4.1}$$

The expressions above are written in the complex coordinates $z_1, z_2, \bar{z}_1, \bar{z}_2$ basis (3.5). The translational collective coordinates of the instanton, $Z_1, Z_2, \bar{Z}_1, \bar{Z}_2$, are c-numbers. Equation (4.1) gives the general solution to the constraints (3.28b),(3.29), and the $U(1)$ instanton moduli space is simply the $\mathbf{R}^4$ which is spanned by $Z_1, Z_2, \bar{Z}_1, \bar{Z}_2$, or equivalently, X_n of (3.32). From now on we will always set these overall translations of the instanton to zero, $Z_1 = Z_2 = \bar{Z}_1 = \bar{Z}_2 = 0$.

The factorization condition (3.15) is then automatically satisfied and

$$f = 1\zeta + z_1\bar{z}_1 + z_2\bar{z}_2 \ . \tag{4.2}$$

The final step in the ADHM set-up is the construction of the normalized matrix U such that $\bar{\Delta}U = 0$ and $\bar{U}U = 1$, as required by (3.12),(3.13), and the expression for the gauge field will follow from (3.14). The (unnormalized) solution U_0 is easy to find:

$$\tilde{U}_0 = \begin{pmatrix} z_1\bar{z}_1 + z_2\bar{z}_2 \\ -\sqrt{\zeta}\bar{z}_2 \\ -\sqrt{\zeta}\bar{z}_1 \end{pmatrix} , \qquad \bar{\Delta}\tilde{U}_0 = 0 \ . \tag{4.3}$$

The problem is that $\tilde{U}_0$ is not straightforwardly normalizable.

4.1. $\mathbf{R}^4_{\mathrm{NC}}$

Let us start with $\mathbf{R}^4_{\mathrm{NC}}$ space with SD θ. It is easy to see that $\tilde{U}_0$ annihilates the vacuum,[9]

[9] For an instanton centered at $(Z_1, Z_2, \bar{Z}_1, \bar{Z}_2)$ the annihilated state will be the corresponding coherent state $|\bar{Z}_1, \bar{Z}_2\rangle$.

$\tilde{U}_0|0,0\rangle = 0$. In order to find the normalized expression for U, the vacuum state $|0,0\rangle$ has to be projected out. An elegant realization of this idea was proposed in [16]. First define a projector $p = 1 - |0,0\rangle\langle 0,0|$. Then the normalized matrix U can be determined via

$$U = \tilde{U}_0 \beta u^\dagger \ , \quad \bar{U}U = 1 \ , \tag{4.4}$$

where β is the normalization factor,

$$\beta = p 1 \sqrt{\tilde{U}_0^\dagger \tilde{U}_0} p = p 1 \sqrt{(z_1\bar{z}_1 + z_2\bar{z}_2)(z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta)} p \ , \tag{4.5}$$

and $u^\dagger$ is a shift operator which projects out the vacuum:

$$\begin{aligned} &u^\dagger \ : \ \mathcal{H} \to p\mathcal{H} \ , \quad u \ : \ p\mathcal{H} \to \mathcal{H} \ , \\ &uu^\dagger = 1 \ , \quad u^\dagger u = p \ , \quad pu^\dagger = u^\dagger \ , \quad up = u \ . \end{aligned} \tag{4.6}$$

Due to the factors of p in the definition of β which projects out the dangerous vacuum state, the right hand side of (4.5) is not singular and well-defined.

It is also straightforward to check that the ADHM completeness relation (3.16) is satisfied, $1 - U\bar{U} = \Delta f \bar{\Delta}$, and the field-strength F_{mn} is in fact anti-selfdual. The topological charge Q can be now calculated as a trace on the Hilbert space. The result is [14, 16]

$$Q = -4 \sum_{(n_1,n_2)\neq 0} 1 (n_1+n_2)(n_1+n_2+1)^2(n_1+n_2+2) = -4 \sum_{N=1}^{\infty} 1 N(N+1)(N+2) = -1 \ , \tag{4.7}$$

in agreement with the general argument of the previous section §3.3 that $|Q| = k$. An alternative calculation from [16] evaluates $Q = -1$ by integrating over the c-number operator symbols.

To summarize: the gauge-field ASD instanton configuration resulting from (4.4), is a local minimum of the action in the $U(1)$ theory on $\mathbf{R}^4_{\mathrm{NC}}$ with the SD-θ. The instanton action is $S_{\mathrm{inst}} = 8\pi^2/g^2$, and the topological charge is $Q = -1$. The instanton configuration is perfectly regular and can be expanded around in the functional integral, leading to quantum instanton contributions in the $U(1)$ theory in the SD-θ background. At the same time the SD $U(1)$ instanton in a SD-θ background does not exist as the corresponding unmodified ADHM constraints have no non-trivial solution for $N = 1$.

4.2. $\mathbf{R}^2_{\mathrm{NC}} \times \mathbf{R}^2$

We now consider $\mathbf{R}^2_{\mathrm{NC}} \times \mathbf{R}^2$ space. The matrix $\tilde{U}_0$ in (4.3) annihilates the vacuum state $|0\rangle$ of $\mathbf{R}^2_{\mathrm{NC}}$ *at the point* $z_2 = 0 = \bar{z}_2$. This is the crucial difference from the unconditional annihilation of a state in the previous example. Let us first try to follow the same route as in §4.1 and normalize U by projecting out the offending state. This projection is necessary if we want to have a normalized U everywhere on the complex plane z_2, including the origin $z_2 = 0 = \bar{z}_2$. We introduce a projector $p = 1 - |0\rangle\langle 0|$, and define the normalized matrix U via (4.4)-(4.6).

A straightforward calculation shows that the ADHM completeness relation (3.16) is not satisfied, and, hence, the field-strength F_{mn} is not anti-selfdual. In fact, let us check the relation, $1 - U\bar{U} = \Delta f \bar{\Delta}$, for the 11-matrix element:

$$
\begin{aligned}
(1 - U\bar{U})_{11} &= 1 - \sum_{n \neq 0} z_1\bar{z}_1 + z_2\bar{z}_2 z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta|n\rangle\langle n| \ , \\
(\Delta f \bar{\Delta})_{11} &= \zeta z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta \ , \\
(1 - U\bar{U})_{11} - (\Delta f \bar{\Delta})_{11} &= z_2\bar{z}_2 z_2\bar{z}_2 + \zeta|0\rangle\langle 0| \ .
\end{aligned}
\tag{4.8}
$$

The last line is non-zero everywhere except at the origin $z_2 = \bar{z}_2 = 0$. This, of course, invalidates the ADHM construction. Since this point is important, it is worthwhile to see more explicitly how the sum of the two orthogonal projectors P and $\mathcal{P}$ fails to span the whole Hilbert space. Consider the normalized state

$$
|\psi\rangle = \begin{pmatrix} |0\rangle \\ 0 \\ 0 \end{pmatrix} . \tag{4.9}
$$

It is easy to check that

$$
\mathcal{P}|\psi\rangle = 0, \quad P|\psi\rangle \neq |\psi\rangle. \tag{4.10}
$$

In fact

$$
P|\psi\rangle = \begin{pmatrix} \zeta(\zeta + z_2\bar{z}_2)^{-1}|0\rangle \\ \sqrt{\zeta}\bar{z}_2(\zeta + z_2\bar{z}_2)^{-1}|0\rangle \\ 0 \end{pmatrix} \neq |\psi\rangle \quad \text{unless } z_2 = 0 = \bar{z}_2. \tag{4.11}
$$

Therefore we see that the orthogonal projectors P and $\mathcal{P}$ do not span the whole Hilbert space.

One might ask what would happen if we did not subtract the vacuum state. In this case the ADHM matrix U will be normalized for all values of z_2 and $\bar{z}_2$ except at the origin $z_2 = 0 = \bar{z}_2$. The resulting gauge field configuration will be singular at $z_2 = 0 = \bar{z}_2$. This singularity is of the form $\phi_{\text{sing}}(x_3, x_4) \cdot |0\rangle\langle 0|$ and is much more severe than the well-known point-like singularity of the commutative instanton in the singular gauge. The c-number operator symbol gauge field will contain a term $\phi_{\text{sing}}(x_3, x_4)e^{-(x_1^2+x_2^2)/\zeta}$, which is singular at $x_3 = 0 = x_4$ on the whole (x_1, x_2)-plane (!), as follows from (2.21). This is not a gauge-removable point-like singularity, and such configurations are not allowed in the semiclassical picture.

4.3. Comments on the ADHM completeness relation

We would like to conclude this section with an analysis of the failure of the ADHM completeness relation (3.16) and the requirement (3.36) derived in the previous section §3.2. Generally the matrix operator $\tilde{U}_0$ can depend on some numerical parameters. These parameters can be either the instanton moduli, or the commutative coordinates as in the $\mathbf{R^2_{\rm NC}} \times \mathbf{R^2}$ example. We will denote collectively these parameters by λ. Now consider a normalized state $|\omega\rangle \in \mathcal{H}$

$$\langle\omega|\omega\rangle = 1, \tag{4.12}$$

which is annihilated by $\tilde{U}_0$ for some specific values, say λ_0, of the parameters. What is important for us here is the fact that $\tilde{U}_0$ does not annihilate $|\omega\rangle$ for all values of the parameters. In order to build a normalizable U we have to project $|\omega\rangle$ out of the Hilbert space on which $\tilde{U}_0$ acts. We again introduce a projector $p = 1 - |\omega\rangle\langle\omega|$, and define the normalized matrix U via (4.4)–(4.6). Note that p is defined for *all* values of λ; therefore $|\omega\rangle$ is a zero-mode of the operator β for *all* λ.

Generally there will also be parameters λ_1 for which $\tilde{U}_0^\dagger \tilde{U}_0$ is invertible. Now fix $\lambda = \lambda_1$ and consider the state $\tilde{U}_0|\omega\rangle$. This state is normalizable and is a zero-mode of P. According to the condition (3.36), the completeness condition (3.16) will hold if there is a state $|s\rangle \in \mathcal{H}$ such that

$$\tilde{U}_0|\omega\rangle \stackrel{?}{=} U|s\rangle = \tilde{U}_0(\beta u^\dagger|s\rangle). \tag{4.13}$$

Obviously this cannot be satisfied since $\tilde{U}_0^\dagger \tilde{U}_0$ is invertible and the state $\beta u^\dagger|s\rangle$ is in $p\mathcal{H}$ while the state $|\omega\rangle$ is in the orthogonal complement $(1-p)\mathcal{H}$.

The above general discussion incorporates the case of the noncommutative space $\mathbf{R^2_{\rm NC}} \times \mathbf{R^2}$ when the parameters discussed above are the commuting coordinates x_3 and x_4. It can also be applied in principle to the other cases where these parameters are some instanton moduli. For example, if this situation is realized the instanton solution will develop singularities for the values of the moduli where $\tilde{U}_0|\omega\rangle = 0$, but will remain regular in the remaining part of the moduli space where $\tilde{U}_0|\omega\rangle \neq 0$.

In §6 we will see that this problem is generic and there is no nonsingular $U(N)$ instanton on $\mathbf{R^2_{\rm NC}} \times \mathbf{R^2}$.

5. U(1) two-instanton solution

In this section we study the ADHM 2-instanton solution in the $U(1)$ gauge theory on $\mathbf{R^4_{\rm NC}}$.

The general 2-instanton solution was first studied in [19]. The ADHM matrix $\bar{\Delta}$ for the ASD instanton (3.11) which satisfies the modified ADHM constraints (3.28b),(3.29) is given by:

$$\bar{\Delta} = \begin{pmatrix} \sqrt{\zeta}\sqrt{1-b} & z_2-\delta_2 & -\delta_2\sqrt{2ba} & z_1-\delta_1 & -\delta_1\sqrt{2ba} \\ \sqrt{\zeta}\sqrt{1+b} & 0 & z_2+\delta_2 & 0 & z_1+\delta_1 \\ 0 & -(\bar{z}_1-\delta_1^*) & 0 & \bar{z}_2-\delta_2^* & 0 \\ 0 & \delta_1^*\sqrt{2ba} & -(\bar{z}_1+\delta_1^*) & -\delta_2^*\sqrt{2ba} & \bar{z}_2+\delta_2^* \end{pmatrix} , \tag{5.1}$$

where δ's are arbitrary c-numbers and

$$a = 2\zeta(|\delta_1|^2+|\delta_2|^2) \ , \quad b = \sqrt{1+a^2}-a \ . \tag{5.2}$$

Equation (5.1) gives the general solution of the constraints (3.28b),(3.29), with the center of mass collective coordinates set to zero, $Z_1 = Z_2 = \bar{Z}_1 = \bar{Z}_2 = 0$. The unconstrained collective coordinates δ_i and δ_i^* give the center of the first instanton; the second instanton is centered at $(-\delta_i, -\delta_i^*)$. The 2-instanton moduli space is 8-dimensional as required, and is spanned by four Z's and four δ's. As shown in [19], after separating the center of mass, the relative moduli space is given by the Eguchi-Hanson manifold (which is non-singular even at the origin where the two point-like $U(1)$ instantons coincide).

In the dilute instanton gas limit $|\delta_i| \to \infty$, the expression on the right hand side of (5.1) clearly splits into two single-instanton expressions:

$$\bar{\Delta}_\infty = \begin{pmatrix} \sqrt{\zeta} & z_2-\delta_2 & 0 & z_1-\delta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -(\bar{z}_1-\delta_1^*) & 0 & \bar{z}_2-\delta_2^* & 0 \\ 0 & 0 & 0 & 0 & \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \\ \sqrt{\zeta} & 0 & z_2+\delta_2 & 0 & z_1+\delta_1 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & -(\bar{z}_1+\delta_1^*) & 0 & \bar{z}_2+\delta_2^* \end{pmatrix} . \tag{5.3}$$

In the opposite limit of coincident instantons, $\delta_i \to 0$, and $a \to 0$, $b \to 1$ in such a way that

$$\delta_1\sqrt{2ba} \to -\sqrt{\zeta}\lambda_1 \ , \quad \delta_2\sqrt{2ba} \to -\sqrt{\zeta}\lambda_2 \ , \quad |\lambda_1|^2+|\lambda_2|^2 = 1 \ . \tag{5.4}$$

Here the λ_1 and λ_2 are the collective coordinate which describe the three angles of the direction in which the two instantons have approached each other. This coincident 2-instanton solution was studied in [15].

We expect that the general 2-instanton ASD configuration is a regular solution to the ASD equation which gives a local minimum of the action $S = 16\pi^2/g^2$ and has topological charge $Q = -2$. Then it can contribute to the functional integral and is semiclassically relevant. The topological charge of the coincident solution was studied in [20] where it was concluded that Q is not integer and is in general moduli-dependent. If true, this would make the instanton action moduli-dependent which would conflict with the statement that instantons are local minima of the action. We have redone the calculation of [20] and found $Q = -2$.

To simplify things a little we fix the angle of approach as in [20], $\lambda_1 = 1$, $\lambda_2 = 0$. The corresponding Δ and $\bar\Delta$ matrices are:

$$\Delta = \begin{pmatrix} 0 & \sqrt{2\zeta} & 0 & 0 \\ \bar z_2 & 0 & -z_1 & -\sqrt{\zeta} \\ 0 & \bar z_2 & 0 & -z_1 \\ \bar z_1 & 0 & z_2 & 0 \\ \sqrt{\zeta} & \bar z_1 & 0 & z_2 \end{pmatrix} , \quad \bar\Delta = \begin{pmatrix} 0 & z_2 & 0 & z_1 & \sqrt{\zeta} \\ \sqrt{2\zeta} & 0 & z_2 & 0 & z_1 \\ 0 & -\bar z_1 & 0 & \bar z_2 & 0 \\ 0 & -\sqrt{\zeta} & -\bar z_1 & 0 & \bar z_2 \end{pmatrix} . \tag{5.5}$$

The unnormalized matrix $\tilde U_0$, which solves $\bar\Delta \tilde U_0 = 0$, is given by

$$\tilde U_0 = \begin{pmatrix} 1\sqrt{2\zeta}((z_1\bar z_1 + z_2\bar z_2)(z_1\bar z_1 + z_2\bar z_2 - \zeta/2) + \zeta z_2\bar z_2) \\ \sqrt{\zeta}\bar z_2\bar z_1 \\ -\bar z_2(z_1\bar z_1 + z_2\bar z_2 + \zeta/2) \\ \sqrt{\zeta}\bar z_1\bar z_1 \\ -\bar z_1(z_1\bar z_1 + z_2\bar z_2 - \zeta/2) \end{pmatrix} . \tag{5.6}$$

This expression annihilates two states: $|0,0\rangle$ and $|1,0\rangle$, which have to be projected out in the usual fashion, (4.4),(4.6), where the projector p is given by $p = 1 - |0,0\rangle\langle 0,0| - |1,0\rangle\langle 1,0|$.

The factorization condition (3.15) is automatically satisfied and

$$f^{-1} = \begin{pmatrix} \zeta + z_1\bar z_1 + z_2\bar z_2 & \sqrt{\zeta}\bar z_1 \\ \sqrt{\zeta} z_1 & 2\zeta + z_1\bar z_1 + z_2\bar z_2 \end{pmatrix} , \tag{5.7}$$

which can be inverted as follows:

$$f = \begin{pmatrix} n_1 + n_2 + 5(n_1 + n_2 + 2)(n_1 + n_2 + 5) - 2(n_1 + 1) & -1(n_1 + n_2 + 2)(n_1 + n_2 + 5) - 2(n_1 + 1)\sqrt{2}\bar z_1 \\ -1(n_1 + n_2 + 4)(n_1 + n_2 + 1) - 2n_1\sqrt{2} z_1 & n_1 + n_2 + 1(n_1 + n_2 + 4)(n_1 + n_2 + 1) - 2n_1 \end{pmatrix} , \tag{5.8}$$

where we have set $\zeta = 2$ and introduced the SHO occupation numbers $n_1 = z_1\bar z_1$ and $n_2 = z_2\bar z_2$.

We can now evaluate the field strength (3.19) and represent the topological charge Q as a trace over $p\mathcal{H}$. We have obtained an analytic expression for the topological charge density identical to the expression in Eq. (31) of [20]. To determine the topological charge Q, we evaluated the corresponding trace by summing over the SHO occupation numbers $(n_1, n_2) \neq (0,0) \neq (1,0)$. We performed this double infinite sum numerically in Maple sampling over 40,000 points (n_1, n_2). Our result is

$$Q \simeq -1.99987 \simeq -2 , \tag{5.9}$$

which is different from the numerical calculation of [20] which gave -0.932.

6. U(N) instantons

6.1. Commutative ASD instanton

Before addressing noncommutative instantons, we first recall the ADHM construction of the standard ASD 1-instanton solution in commutative $U(N)$. The ASD 1-instanton is determined from the ADHM matrices (3.11) subject to the constraints (3.28b) and (3.29) with $\zeta \equiv 0$. Eq. (3.28b) says that a'_n is real,

$$a'_n \;\equiv\; -X_n \;\in\; \mathbf{R}^4 \;, \tag{6.1}$$

after which Eq. (3.29) with $\zeta \equiv 0$ collapses to

$$\bar{w}_u^{\dot{\alpha}}\, w_{u\dot{\beta}} \;=\; \rho^2\, \delta^{\dot{\alpha}}_{\;\dot{\beta}} \;. \tag{6.2}$$

The quantities ρ and X_n will soon be identified with the instanton scale size and space-time position, respectively. It is convenient to put w in the form:

$$w_{[N]\times[2]} \;=\; \rho\,\mathcal{U}_{[N]\times[N]} \begin{pmatrix} 0_{[N-2]\times[2]} \\ 1_{[2]\times[2]} \end{pmatrix} \;, \qquad \mathcal{U} \in U(N)U(1)U(N-2) \;. \tag{6.3}$$

Setting $\mathcal{U} = 1$ initially, we find for Δ and f :

$$\Delta_{[N+2]\times[2]} \;=\; \begin{pmatrix} 0_{[N-2]\times[2]} \\ \rho\cdot 1_{[2]\times[2]} \\ y_{[2]\times[2]} \end{pmatrix} \;, \qquad f \;=\; 1y^2 + \rho^2 \;, \tag{6.4}$$

with $y = x - X$. We now solve (3.12), (3.13) and determine the normalized matrix U:

$$U_{[N+2]\times[N]} \;=\; \begin{pmatrix} 1_{[N-2]\times[N-2]} & 0 \\ 0 & \left(y^2y^2+\rho^2\right)^{1/2} 1_{[2]\times[2]} \\ 0_{[2]\times[N-2]} & -\left(\rho^2 y^2 (y^2+\rho^2)\right)^{1/2} \bar{y}_{[2]\times[2]} \end{pmatrix} \;. \tag{6.5}$$

The gauge field then follows from Eq. (3.14):

$$A_n \;=\; \begin{pmatrix} 0 & 0 \\ 0 & A_n^{SU(2)} \end{pmatrix} \;. \tag{6.6}$$

Here $A_n^{SU(2)}$ is the standard singular-gauge $SU(2)$ anti-instanton [37,38] with space-time position X_n , size ρ and in a fixed iso-orientation:

$$A_n^{SU(2)}(x) \;=\; i\rho^2\;\eta^a_{nm}\;(x-X)^m\;\tau^a(x-X)^2\;\left((x-X)^2+\rho^2\right) \;. \tag{6.7}$$

For a general iso-orientation matrix $\mathcal{U}$ we obtain instead

$$A_n \;=\; \mathcal{U}\begin{pmatrix} 0 & 0 \\ 0 & A_n^{SU(2)} \end{pmatrix}\mathcal{U}^\dagger\;, \qquad \mathcal{U}\in U(N)U(1)\times U(N-2)\;. \tag{6.8}$$

We see that the instanton always lives in an $SU(2)$ subgroup of the $SU(N)$ gauge group. An explicit representation of this embedding is formed by the three composite $SU(2)$ generators

$$(t^c)_{uv} \;=\; \rho^{-2}\, w_{u\dot\alpha}\,(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\bar w^{\dot\beta}_v, \qquad c=1,2,3. \tag{6.9}$$

6.2. ASD instanton in the SD background

The ASD 1-instanton in the SD-θ background in the noncommutative $U(N)$ theory is characterized by the ADHM matrices (3.11) subject to the constraints (3.28b) and (3.29). For the case at hand with $k=1$ these constraints are solved by choosing the $N\times 2$ matrix w in (3.23a) in the form:

$$w_{[N]\times[2]} = \mathcal{U}_{[N]\times[N]}\begin{pmatrix} 0_{[N-2]} & 0_{[N-2]} \\ 0 & \rho \\ \sqrt{\zeta+\rho^2} & 0 \end{pmatrix}\;, \quad \mathcal{U}\in U(N)U(1)U(N-2)\;, \tag{6.10}$$

where ρ denotes the instanton size, and $\mathcal{U}_{[N]\times[N]}$ specifies the embedding of the $U(2)$ subgroup into the gauge group $U(N)$. The expression in (6.10) gives the general solution of the ADHM constraints. It follows from (6.10) that for $N\geq 2$ the $U(N)$ noncommutative instantons are essentially given by the $U(2)$ noncommutative instantons. The fact that the building blocks for the noncommutative instantons gauge fields are the 2×2 matrices in the group space, just as in the ordinary commutative case, is non-trivial. One might have expected that, since there are noncommutative $U(1)$ instantons, two types of building blocks for noncommutative $U(N)$ could exist: $U(2)$-instantons and $U(1)$-instantons. We will see that the $U(1)$-instanton building blocks appear *inside* the $U(2)$ blocks when the instanton size ρ shrinks to zero.

To keep expressions simple, from now on we set $\mathcal{U}_{[N]\times[N]}=1$. In this case the instanton positioned at the origin is determined from:

$$\Delta = \begin{pmatrix} 0_{[N-2]} & 0_{[N-2]} \\ 0 & \rho \\ \sqrt{\zeta+\rho^2} & 0 \\ \bar z_2 & -z_1 \\ \bar z_1 & z_2 \end{pmatrix}\;, \qquad \bar\Delta = \begin{pmatrix} 0_{[N-2]} & 0 & \sqrt{\zeta+\rho^2} & z_2 & z_1 \\ 0_{[N-2]} & \rho & 0 & -\bar z_1 & \bar z_2 \end{pmatrix}\;. \tag{6.11}$$

Since the instanton configuration is concentrated in a $U(2)$ factor of the gauge group, we will set $N=2$ from now on. The factorization relation follows and

$$f = 1z_1\bar z_1 + z_2\bar z_2 + \rho^2 + \zeta\;. \tag{6.12}$$

There are two interesting special cases to notice. First, when $\zeta \to 0$, Eqs. (6.11)-(6.12) collapse to the defining equations for the ordinary commutative BPST instanton (see section 2.3 of [32] for a review). On the other hand, we can consider the limit $\rho \to 0$, with ζ fixed. In this case Eqs. (6.11)-(6.12) collapse essentially to the $U(1)$ instanton case (4.1)-(4.2). Thus we conclude that the regular $U(1)$ instantons arise in the limit of instanton sizes going to zero.

An ansatz for the unnormalized matrix $\tilde{U}_0$ was given in [15]:

$$\tilde{U}_0 = \begin{pmatrix} 0 & (z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta)\sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta} \\ z_1\bar{z}_1 + z_2\bar{z}_2 & 0 \\ -\sqrt{\zeta+\rho^2}\bar{z}_2 & \rho z_1\sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta} \\ -\sqrt{\zeta+\rho^2}\bar{z}_1 & -\rho z_2\sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta} \end{pmatrix} , \qquad \bar{\Delta}\tilde{U}_0 = 0 \ . \tag{6.13}$$

As in the $U(1)$ case, it is easy to see that $\tilde{U}_0$ annihilates the vacuum $\tilde{U}_0|0,0\rangle = 0$, which has to be projected out as in (4.4),(4.6). The normalization factor is given by

$$\beta = p 1\sqrt{\tilde{U}_0^\dagger \tilde{U}_0 p} = p\ 1_{[2]\times[2]} 1\sqrt{(z_1\bar{z}_1 + z_2\bar{z}_2)(z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta + \rho^2)p} \ . \tag{6.14}$$

However it is easy to check that a U of the form (4.4) used in [15]

$$U = \tilde{U}_0 \beta u^\dagger \ , \tag{6.15}$$

does not satisfy the completeness condition. To see this, let us first give the expression $\Delta f \Delta^\dagger$. It is

$$\Delta f \bar{\Delta} = \begin{pmatrix} \rho^2 f & 0 & -\rho f \bar{z}_1 & \rho f \bar{z}_2 \\ 0 & (\zeta+\rho^2) f & \sqrt{\zeta+\rho^2} f z_2 & \sqrt{\zeta+\rho^2} f z_1 \\ -\rho z_1 f & \sqrt{\zeta+\rho^2}\bar{z}_2 f & \bar{z}_2 f z_2 + z_1 f \bar{z}_1 & \bar{z}_2 f z_1 - z_1 f \bar{z}_2 \\ \rho z_2 f & \sqrt{\zeta+\rho^2}\bar{z}_1 f & \bar{z}_1 f z_2 - z_2 f \bar{z}_1 & \bar{z}_1 f z_1 + z_2 f \bar{z}_2 \end{pmatrix} . \tag{6.16}$$

Now acting on the state $\langle 0,0|$, we have

$$\langle 0,0|UU^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} . \tag{6.17}$$

Obviously the completeness relation is not satisfied! Notice that an ansatz of the form (4.4), where there is an overall factor of the shift operator $u^\dagger$, works well for the case of $U(1)$. There is no reason to restrict oneself to this form[10] for higher $U(N)$. Indeed a more general solution

[10] In fact it is easy to find a zero mode of P which cannot be written as $\tilde{U}_0(\beta u^\dagger|s\rangle)$. This zero mode is given by $U_{\rm new}\begin{pmatrix} 0 \\ |00\rangle \end{pmatrix}$, where $U_{\rm new}$ refers to the right hand side of Eq. (6.18).

which does not take this form can be written down,

$$U = \begin{pmatrix} 0 & \sqrt{\frac{z_1\bar{z}_1+z_2\bar{z}_2+\zeta}{z_1\bar{z}_1+z_2\bar{z}_2+\rho^2+\zeta}} \\ (z_1\bar{z}_1 + z_2\bar{z}_2)\beta u^\dagger & 0 \\ -\sqrt{\zeta+\rho^2}\bar{z}_2\beta u^\dagger & \rho z_1 \frac{1}{\sqrt{(z_1\bar{z}_1+z_2\bar{z}_2+\zeta)(z_1\bar{z}_1+z_2\bar{z}_2+\rho^2+\zeta)}} \\ -\sqrt{\zeta+\rho^2}\bar{z}_1\beta u^\dagger & -\rho z_2 \frac{1}{\sqrt{(z_1\bar{z}_1+z_2\bar{z}_2+\zeta)(z_1\bar{z}_1+z_2\bar{z}_2+\rho^2+\zeta)}} \end{pmatrix}. \tag{6.18}$$

A special feature of this solution is that the shift operator $u^\dagger$ appears only in the first column of U, where β appears. It is not difficult to show that this is indeed a general feature for the form of U in the nonabelian case.

It is easy to check that (6.18) satisfies

$$\bar{\Delta}U = 0, \quad \bar{U}U = 1_{[2]\times[2]}. \tag{6.19}$$

Moreover one can check that the ADHM completeness relation (3.16)

$$1 - U\bar{U} = \Delta f \bar{\Delta} \tag{6.20}$$

is satisfied, and the field-strength F_{mn} is anti-selfdual.

We further note that at large distances $z_i\bar{z}_i \gg \zeta$, (6.18) coincides with the matrix U of the commutative instanton (6.5). It follows that at distances large compared to noncommutativity scale, the noncommutative instanton gauge field coincides with the commutative instanton in the singular gauge (6.6),(6.7). On the other hand, at short distances the noncommutativity parameter ζ regulates the short-distance singularity in (6.7). Hence we get a regular noncommutative ASD instanton which at large distances looks like the BPST instanton in the singular gauge. This means that the LSZ reduction formulae can be applied as usual to the functional integral representation of the instanton Green functions for calculating instanton contributions to effective actions (see e.g. [34]).

Finally we calculate the topological charge Q. Using (3.10) and (3.19), it is easy to obtain

$$\mathrm{Tr}_N F^2 = -16\mathrm{Tr}(A^2 + D^2 + 4AD - 2BC)f^2, \tag{6.21}$$

where we have denoted

$$\bar{b}U\bar{U}b = \begin{pmatrix} (\rho^2+\zeta)\bar{z}_2\beta^2 z_2 + \rho^2 z_1 f g \bar{z}_1 & (\rho^2+\zeta)\bar{z}_2\beta^2 z_1 - \rho^2 z_1 f g \bar{z}_2 \\ (\rho^2+\zeta)\bar{z}_1\beta^2 z_2 - \rho^2 z_2 f g \bar{z}_1 & (\rho^2+\zeta)\bar{z}_1\beta^2 z_1 + \rho^2 z_2 f g \bar{z}_2 \end{pmatrix} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6.22}$$

and

$$g := \frac{1}{z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta}. \tag{6.23}$$

Substituting (6.21) into the definition (3.2), we find

$$Q = -\sum_{N=1}^{\infty} 4\,(N+1)N(N+2a^2+2)^2(N+2a^2+1)^2(N+2)(N+2a^2+3)^2(N+2a^2)$$
$$\times\,(748a^4N^2+144a^2N+350a^2N^2+332a^4N+230a^2N^3+62a^2N^4+432a^6N+272a^8N$$
$$+\,716\,a^6N^2+304a^8N^2+406a^4N^3+280a^6N^3+84a^4N^4+48a^{10}N^2$$
$$+\,72a^8N^3+64a^{10}N+6a^2N^5+6a^4N^5+36a^6N^4+36N-296a^6-160a^8$$
$$-\,32a^{10}+37N^3-72a^2-240a^4+60N^2+N^5+10N^4) \;=\; -1\;, \tag{6.24}$$

where $a=\rho/\sqrt{\zeta}$, and $N=n_1+n_2$. Note that the dependence on the instanton modulus ρ disappears in the final answer. Here we disagree with the results of [20] which reported ρ-dependence in Q for the case at hand. Our result (6.24), as expected, is in agreement with general argument of §3.3 that $|Q|=k$.

6.3. No instantons in $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}}\times\mathbf{R}^{\mathbf{2}}$

It is easy to see that just as in the $U(1)$ case discussed earlier in §4.2 and §4.3, one faces a conceptual problem in trying to construct a $U(N)$ instanton on $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}}\times\mathbf{R}^{\mathbf{2}}$. The argument runs as follows. First, we have to subtract the vacuum state $|0\rangle$ in order to avoid singularities[11] and introduce the shift operator $u^\dagger$. Now observe that $uu^\dagger=1$ and rewrite the matrix U in (6.18) as

$$U=\begin{pmatrix} 0 & \sqrt{\frac{z_1\bar z_1+z_2\bar z_2+\zeta}{z_1\bar z_1+z_2\bar z_2+\rho^2+\zeta}}\,u \\ (z_1\bar z_1+z_2\bar z_2)\beta & 0 \\ -\sqrt{\zeta+\rho^2}\,\bar z_2\beta & \rho z_1\frac{1}{\sqrt{(z_1\bar z_1+z_2\bar z_2+\zeta)(z_1\bar z_1+z_2\bar z_2+\rho^2+\zeta)}}\,u \\ -\sqrt{\zeta+\rho^2}\,\bar z_1\beta & -\rho z_2\frac{1}{\sqrt{(z_1\bar z_1+z_2\bar z_2+\zeta)(z_1\bar z_1+z_2\bar z_2+\rho^2+\zeta)}}\,u \end{pmatrix} u^\dagger \equiv \tilde U u^\dagger\;. \tag{6.25}$$

For values of $z_2\neq 0$ we can construct a zero mode of P which does not satisfy the condition (3.36):

$$\tilde U\begin{pmatrix}|0\rangle\\ 0\end{pmatrix} \stackrel{?}{=} U\begin{pmatrix}|s_1\rangle\\ |s_2\rangle\end{pmatrix} = \tilde U\begin{pmatrix}u^\dagger|s_1\rangle\\ u^\dagger|s_2\rangle\end{pmatrix}\;. \tag{6.26}$$

This cannot be satisfied since $\tilde U^\dagger\tilde U$ is invertible and the state $u^\dagger|s_1\rangle$ is in $p\mathcal{H}$ while the state $|0\rangle$ is in the orthogonal complement $(1-p)\mathcal{H}$. Hence, the (anti)-self-duality of the field strength is violated and there is no nonsingular $U(N)$ instanton on $\mathbf{R}^{\mathbf{2}}_{\mathrm{NC}}\times\mathbf{R}^{\mathbf{2}}$.

[11] The singularities would occur on the plane at $z_2=0=\bar z_2$, see the last paragraph in §4.2.

Instantons and vacuum tunneling

It is well-known that instanton field configurations in ordinary commutative gauge theories correspond to paths that connect initial and final vacuum states with different winding numbers [40, 41]. The expression (6.7)

$$A_n(x) \; = \; i\rho^2 \; \eta^a_{nm} \; x^m \; \tau^a x^2 \; (x^2+\rho^2) \; , \tag{6.27}$$

is singular as $|x|^2 \to 0$ and is known as the ASD instanton in the singular gauge. It can be obtained from the regular-gauge ASD instanton,

$$A_n = i \; \bar{\eta}^a_{nm} \; x^m \; \tau^a x^2 + \rho^2 \; , \tag{6.28}$$

with a singular $SU(2)$ gauge transformation, $S^\dagger = x_m \sigma^m / \sqrt{x^2}$. Hence the singularity on the right hand side of (6.27) is just a gauge artifact. One can see how the commutative instanton configuration interpolates between two different semiclassical vacua at $x_4 \to \pm\infty$. First one has to gauge transform the regular-gauge instanton (6.28) into the $A'_4 = 0$ gauge, and then it follows that

$$A'_i\,(\vec{x}, x_4 = \pm\infty) \; = \; V_\pm(\vec{x})\, \partial_i \, V^\dagger_\pm(\vec{x}) \; , \tag{6.29}$$

where

$$\begin{aligned} V_-(\vec{x}) &= \exp\left(-i\pi n \vec{x}\cdot\vec{\tau}\sqrt{\vec{x}^2+\rho^2}\right) \; , \\ V_+(\vec{x}) &= \exp\left(-i\pi(n+1)\vec{x}\cdot\vec{\tau}\sqrt{\vec{x}^2+\rho^2}\right) \; , \end{aligned} \tag{6.30}$$

and n is an arbitrary integer. Hence, the instanton is a trajectory in the configuration space, $A_i(\vec{x})$, parametrized by the Euclidean time coordinate x_4. The two ends of this trajectory, $x_4 \to \pm\infty$, are the semiclassical vacuum configurations, (6.29), which are pure gauges with the winding numbers n and $n+1$.

In order to apply this argument to noncommutative gauge theories we need to find an instanton trajectory parametrized by a c-number parameter x_4. Hence we need regular instantons on $\mathbf{R}^{\mathbf{2}}_{\rm NC} \times \mathbf{R}^{\mathbf{2}}$. Since these instanton configurations cannot be found we are tempted to conclude that the vacuum structure of noncommutative $U(N)$ theories is trivial for all values of $N \geq 1$. This conclusion is somewhat unexpected, since time-independent gauge transformations on $\mathbf{R}^{\mathbf{2}}_{\rm NC} \times \mathbf{R}^{\mathbf{2}}$ which approach the identity at infinity are supposed to be [42] the maps from R^1 to $U_{\rm compact}(\mathcal{H})$, or equivalently maps from S^1 to $U_{\rm compact}(\mathcal{H})$. Here $R^1 \sim S^1$ is spanned by the x_3 commutative coordinate, and $U_{\rm compact}(\mathcal{H})$ denotes the unitary operators over $\mathcal{H}$ of the form $U = 1 + K$, where K is a compact operator. This map is non-trivial, and in particular, $\pi_1(U_{\rm compact}) = \mathbf{Z}$. This suggests a non-trivial vacuum structure parametrized by an integer winding number for all values of N. Whether this is true or not, we cannot find instanton trajectories which would interpolate between. Perhaps the map above, $S^1 \to U_{\rm compact}(\mathcal{H})$, is too restrictive and one should use instead, $S^1 \to U(\mathcal{H})$, which is contractible and so has trivial topology.

6.4. SD instanton in the SD background

Here we investigate the SD 1-instanton solution of the $U(N)$ noncommutative theory in the SD-θ background. This solution has been previously studied in [17].

Without a loss of generality, we specialize here to the minimal case of gauge group $U(2)$. The matrix Δ is determined by solving the unmodified ADHM constraints (3.26) and reads:

$$\Delta = \begin{pmatrix} \rho & 0 \\ 0 & \rho \\ z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix} , \qquad \bar{\Delta} = \begin{pmatrix} \rho & 0 & \bar{z}_2 & -z_1 \\ 0 & \rho & \bar{z}_1 & z_2 \end{pmatrix} . \tag{6.31}$$

The factorization relation follows, and

$$f = 1 z_1\bar{z}_1 + z_2\bar{z}_2 + \rho^2 + \zeta/2 . \tag{6.32}$$

The normalized matrix U can now be constructed directly:

$$U = \begin{pmatrix} -\bar{z}_2\sqrt{1 z_1\bar{z}_1 + z_2\bar{z}_2 + \rho^2} & z_1\sqrt{1 z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta + \rho^2} \\ -\bar{z}_1\sqrt{1 z_1\bar{z}_1 + z_2\bar{z}_2 + \rho^2} & -z_2\sqrt{1 z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta + \rho^2} \\ \rho\sqrt{1 z_1\bar{z}_1 + z_2\bar{z}_2 + \rho^2} & 0 \\ 0 & \rho\sqrt{1 z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta + \rho^2} \end{pmatrix} , \quad \bar{\Delta}U = 0 , \quad \bar{U}U = 1 . \tag{6.33}$$

Note that in the SD/SD case at hand there are no states annihilated by U and no shifts operators are introduced in (6.33) in distinction with the ASD/SD cases considered earlier.

One can check with a straightforward calculation that the ADHM completeness relation (3.16) is satisfied, $1 - U\bar{U} = \Delta f \bar{\Delta}$, and that the field-strength F_{mn} is selfdual.

Now we want to investigate the behaviour of the SD instanton at large distances, $z_i\bar{z}_i \gg \zeta$, where noncommutativity can be neglected. It follows from (6.33) that in this limit the non-commutative instanton coincides with the commutative BPST instanton in the *regular* gauge,

$$A_n = \bar{U}\partial_n U \;\rightarrow\; i\,\bar{\eta}^a_{nm}\, x^m\, \tau^a x^2 + \rho^2 . \tag{6.34}$$

It is interesting to compare this SD/SD instanton with the ASD/SD solution discussed earlier. While the former approaches the regular-gauge BPST instanton, the latter tends to the singular-gauge BPST anti-instanton. One might wonder if it is possible to gauge-transform the SD/SD instanton in such a way that it tends to the singular-gauge BPST SD instanton.

In the commutative set-up one can always pass from the regular-gauge SD instanton to the singular gauge SD instanton with a singular $SU(2)$ gauge transformation, $S^\dagger = x_m \sigma^m / \sqrt{x^2}$. The noncommutative generalization of $S^\dagger$ is

$$S^\dagger = \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix} 1 \sqrt{z_1 \bar{z}_1 + z_2 \bar{z}_2 + \zeta/2} \ . \tag{6.35}$$

It is easy to see that this is not a unitary operator,

$$SS^\dagger = 1 \ , \quad S^\dagger S = \begin{pmatrix} 1 - |0,0\rangle\langle 0,0| & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{6.36}$$

hence $S^\dagger$ is not an allowed gauge transformation on the Hilbert space $\mathcal{H}$.

The LSZ reduction formulae cannot be applied directly to the gauge field component A_n of the SD/SD instanton, since it does not fall off sufficiently fast at large distances. However the LSZ amputation rules can still be applied to the field strength and to the scalar-field and fermion-field components of the instanton supermultiplet. This is all what is required in e.g. deriving instanton contributions to the Seiberg-Witten prepotential in the commutative [34] and the noncommutative case [11].

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